

OPTIMAL CONTROL AND NONLINEAR FILTERING
FOR NONDEGENERATE DIFFUSION PROCESSES

by

Wendell H. Fleming*
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

and

Sanjoy K. Mitter**
Department of Electrical Engineering
and Computer Science
and
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

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1. Introduction We consider an n -dimensional signal process $x(t) = (x_1(t), \dots, x_n(t))$ and a 1-dimensional observation process $y(t)$, obeying the stochastic differential equations

$$(1.1) \quad dx = b[x(t)]dt + \sigma[x(t)]dw$$

$$(1.2) \quad dy = h[x(t)]dt + d\tilde{w}, \quad y(0) = 0,$$

with w, \tilde{w} independent standard brownian motions of respective dimensions $n, 1$. (The extensions to vector-valued $y(t)$ need only minor modifications.) The Zakai equation for the unnormalized conditional density $q(x, t)$ is

$$(1.3) \quad dq = A^* q dt + h q dy, \quad t \geq 0,$$

where A is the generator of the signal process $x(t)$. See [3] for example. By formally substituting

$$(1.4) \quad q(x, t) = \exp [y(t)h(x)]p(x, t)$$

one gets instead of the stochastic partial differential equation (1.3) a linear partial differential equation of the form

$$(1.5) \quad p_t = \frac{1}{2} \text{tr } a(x) p_{xx} + g^y(x, t) \cdot p_x + V^y(x, t) p, \quad t \geq 0,$$

with $p(x, 0) = p^0(x)$ the density of $x(0)$. Here

$$a(x) = \sigma(x)\sigma(x)', \quad p_x = (p_{x_1}, \dots, p_{x_n})$$

$$\text{tr } a(x) p_{xx} = \sum_{i,j=1}^n a_{ij}(x) p_{x_i x_j}.$$

Explicit formulas for g^y, V^y are given in §6. Equation (1.5) is the basic equation of the pathwise theory of nonlinear filtering. See [2] or

[9]. The superscript y indicates dependence on the observation trajectory $y = y(\cdot)$. Of course, the solution $p = p^y$ also depends on y .

We shall impose in (1.1) the nondegeneracy condition that the $n \times n$ matrix $\sigma(x)$ has a bounded inverse $\sigma^{-1}(x)$. Other assumptions on b , σ , h , p^0 will be stated later. Certain unbounded functions h are allowed in the observation equation (1.2). For example, h can be a polynomial in $x = (x_1, \dots, x_n)$ such that $|h(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$.

The connection between filtering and control is made by considering the function $S = -\log p$. This logarithmic transformation changes (1.5) into a nonlinear partial differential equation for $S(x, t)$, of the form (2.2) below. We introduce a certain optimal stochastic control problem for which (2.2) is the dynamic programming equation.

In §3 upper estimates for $S(x, t)$ as $|x| \rightarrow \infty$ are obtained, by using an easy Verification Theorem and suitably chosen comparison controls. Note that an upper estimate for S gives a lower estimate for $p = -\log S$. A lower estimate for $S(x, t)$ as $|x| \rightarrow \infty$ is obtained in §5 by another method from a corresponding upper estimate for $p(x, t)$. These results are applied to the pathwise nonlinear filter equation in §6.

2. The logarithmic transformation. Let us consider a linear parabolic partial differential equation of the form

$$(2.1) \quad p_t = \frac{1}{2} \text{tr } a(x) p_{xx} + g(x, t) \cdot p_x + V(x, t) p, \quad t \geq 0,$$

$$p(x, 0) = p^0(x).$$

When $g = g^y$, $V = V^y$ this becomes the pathwise filter equation (1.5), to

which we return in §6. By solution $p(x,t)$ to (2.1) we mean a "classical" solution $p \in C^{2,1}$, i.e. with p_{x_i} , $p_{x_i x_j}$, p_t continuous, $i, j = 1, \dots, n$.

If p is a positive solution to (2.1), then $S = -\log p$ satisfies the nonlinear parabolic equation

$$(2.2) \quad S_t = \frac{1}{2} \operatorname{tr} a(x) S_{xx} + H(x, t, S_x), \quad t \geq 0.$$

$$S(x, 0) = S^0(x) = -\log p^0(x),$$

$$H(x, t, S_x) = g(x, t) \cdot S_x - \frac{1}{2} S_x' a(x) S_x - V(x, t).$$

Conversely, if $S(x, t)$ is a solution to (2.2), then $p = \exp(-S)$ is a solution to (2.1).

This logarithmic transformation is well known. For example, if $g = V = 0$, then it changes the heat equation into Burger's equation [8].

We consider $0 \leq t \leq t_1$, with t_1 fixed but arbitrary. Let $Q = \mathbb{R}^n \times [0, t_1]$. We say that a function ϕ with domain Q is of class \mathcal{L} if ϕ is continuous and, for every compact $K \in \mathbb{R}^n$, $\phi(\cdot, t)$ satisfies a uniform Lipschitz condition on K for $0 \leq t \leq t_1$. We say that ϕ satisfies a polynomial growth condition of degree r , and write $\phi \in \mathcal{P}_r$, if there exists M such that

$$|\phi(x, t)| \leq M(1 + |x|^r), \quad \text{all } (x, t) \in Q.$$

Throughout this section and §3 the following assumptions are made.

Somewhat different assumptions are made in §'s 4, 5 as needed. We assume:

$$(2.3) \quad \sigma, \sigma^{-1} \text{ are bounded, Lipschitz functions on } \mathbb{R}^n.$$

For some $m \geq 1$

$$(2.4) \quad g \in \mathcal{L} \cap \mathcal{P}_m, \quad v \in \mathcal{L} \cap \mathcal{P}_{2m}.$$

For some $\ell \geq 0$

$$(2.5) \quad s^0 \in C^2 \cap \mathcal{P}_\ell.$$

For some M_1 ,

$$(2.6) \quad V(x,t) \leq M_1, \quad s^0(x) \geq -M_1.$$

We introduce the following stochastic control problem, for which (2.2) is the dynamic programming equation. The process $\xi(t)$ being controlled is n -dimensional and satisfies

$$(2.7) \quad d\xi = \underline{u}(\xi(\tau), \tau) d\tau + \sigma[\xi(\tau)] dw, \quad 0 \leq \tau \leq t,$$

$$\xi(0) = x.$$

The control is feedback, R^n -valued:

$$(2.8) \quad u(\tau) = \underline{u}(\xi(\tau), \tau).$$

Thus, the control \underline{u} is just the drift coefficient in (2.7). We admit any \underline{u} of class $\mathcal{L} \cap \mathcal{P}_1$. Note that $\underline{u} \in \mathcal{P}_1$ implies at most linear growth of $|\underline{u}(x,t)|$ as $|x| \rightarrow \infty$. For every admissible \underline{u} , equation (2.7) has a pathwise unique solution ξ such that $E\|\xi\|_t^r < \infty$ for every $r > 0$. Here $\|\cdot\|_t$ is the sup norm on $[0,t]$.

Let

$$(2.9) \quad L(x,t,u) = \frac{1}{2} (u-g(x,t))' a^{-1}(x) (u-g(x,t)) - V(x,t).$$

For $(x, t) \in Q$ and \underline{u} admissible, let

$$(2.10) \quad J(x, t, \underline{u}) = E_x \left\{ \int_0^t L[\xi(\tau), t-\tau, u(\tau)] d\tau + S^0[\xi(t)] \right\}$$

The polynomial growth conditions in (2.4), (2.5) imply finiteness of J . The stochastic control problem is to find \underline{u}^{op} minimizing $J(x, t, \underline{u})$. Under the above assumptions, we cannot claim that an admissible \underline{u}^{op} exists minimizing $J(x, t, \underline{u})$. However, we recall from [7, Thm. VI 4.1] the following result, which is a rather easy consequence of the Ito differential rule.

Verification Theorem. Let S be a solution to (2.2) of class $C^{2,1} \cap \mathcal{P}_r$, with $S(x, 0) = S^0(x)$. Then

- (a) $S(x, t) \leq J(x, t; \underline{u})$ for all admissible \underline{u} .
- (b) If $\underline{u}^{op} = g - aS_x$ is admissible, then $S(x, t) = J(x, t; \underline{u}^{op})$.

In §3 we use (a) to get upper estimates for $S(x, t)$, by choosing judiciously comparison controls. For \underline{u}^{op} to be admissible, in the sense we have defined admissibility, $|S_x|$ can grow at most linearly with $|x|$; hence $S(x, t)$ can grow at most quadratically. By enlarging the class of admissible controls to include certain \underline{u} with faster growth as $|x| \rightarrow \infty$ one could generalize (b). However, we shall not do so here, since only part (a) will be used in §3 to get an estimate for S .

In §4 we consider the existence of a solution S with the polynomial growth condition required in the Verification Theorem.

As in [6] we call a control problem with dynamics (2.7) a problem of

stochastic calculus of variations. The control $\underline{u}(\xi(\tau), \tau)$ is a kind of "average" time-derivative of $\xi(\tau)$, replacing the nonexistent derivative $\dot{\xi}(\tau)$ which would appear in the corresponding calculus of variations problem with $\sigma = 0$.

Other control problems. There are other stochastic control problems for which (2.2) is also the dynamic programming equation. One choice, which is appealing conceptually, is to require instead of (2.7) that $\xi(\tau)$ satisfy

$$(2.11) \quad d\xi = \left\{ g[\xi(\tau), \tau] + u[\xi(\tau), \tau] \right\} d\tau + \sigma[\xi(\tau)] dw$$

with $\xi(0) = x$. We then take

$$(2.12) \quad L(x, t, u) = \frac{1}{2} u' a^{-1}(x) u - V(x, t).$$

The feedback control \underline{u} changes the drift in (2.11) from g to $g + \underline{u}$. When $a = \text{identity}$, $L = \frac{1}{2} |u|^2 - V(x, t)$ corresponds to an action integral in classical mechanics with time-dependent potential $V(x, t)$.

3. Upper estimates for $S(x, t)$. In this section we obtain the following upper estimates for the growth of $S(x, t)$ as $|x| \rightarrow \infty$ in terms of the constants $m \geq 1, \ell \geq 0$ in (2.4), (2.5).

Theorem 3.1 Let S be a solution of (2.2) of class $C^{2,1} \cap \mathcal{P}_r$, with $S(x, 0) = S^0(x)$. Then there exist positive M_1, M_2 such that:

- (i) For $(x, t) \in Q$, $S(x, t) \leq M_1(1 + |x|^\rho)$ with $\rho = \max(m+1, \ell)$.
- (ii) Let $0 < t_0 < t_1$, $m > 1$. For $(x, t) \in \mathbb{R}^n \times [t_0, t_1]$,
 $S(x, t) \leq M_2(1 + |x|^{m+1})$.

In the hypotheses of this theorem, $S(x,t)$ is assumed to have polynomial growth as $|x| \rightarrow \infty$ with some degree r . The theorem states that r can be replaced by ρ , or indeed by $m+1$ provided $t \geq t_0 > 0$. Purely formal arguments suggest that $m+1$ is best possible, and this is confirmed by the lower estimate for $S(x,t)$ made in §5.

Proof of Theorem 3.1. We first consider $m > 1$. By (2.3)-(2.6) and (2.9),

$$(3.1) \quad \begin{aligned} L(x,t,u) &\leq B_1(1+|x|^{2m}+|u|^2) \\ S^0(x) &\leq B_1(1+|x|^{\ell}) \end{aligned}$$

for some B_1 . Given $x \in \mathbb{R}^n$ we choose the following open loop control $u(\tau)$, $0 \leq \tau \leq t$. Let $u(\tau) = \eta(\tau)$, where the components $\eta_i(\tau)$, satisfy the differential equation

$$(3.2) \quad \dot{\eta}_i = -(\operatorname{sgn} x_i) |\eta_i|^m \quad i = 1, \dots, n,$$

with $\eta(0) = x$. From (2.7)

$$\xi(\tau) = \eta(\tau) + \zeta(\tau), \quad 0 \leq \tau \leq t,$$

$$\zeta(\tau) = \int_0^\tau \sigma[\xi(\theta)] dw(\theta).$$

Since σ is bounded, $E\|\zeta\|_t^r < \infty$ for each r . By explicitly integrating

(3.2) we find, since $m > 1$, that

$$\begin{aligned} \int_0^t |\eta_i(\tau)|^{2m} d\tau &\leq \frac{1}{m+1} |x_i|^{m+1} \leq \frac{1}{m+1} |x|^{m+1}, \\ E \int_0^t |\xi(\tau)|^{2m} d\tau &\leq 2^{2m} \left[\int_0^t |\eta(\tau)|^{2m} d\tau + E \int_0^t |\zeta(\tau)|^{2m} d\tau \right] \leq M_3(1+|x|^{m+1}) \end{aligned}$$

for some M_3 .

$$\text{Since } u_i^2 = \dot{\eta}_i^2 = \eta_i^{2m},$$

$$\int_0^t |u(\tau)|^2 d\tau \leq \frac{n}{m+1} |x|^{m+1}.$$

Since $|\eta(t)| \leq |x|$,

$$E|\xi(t)|^\ell \leq E(|x| + |\zeta(t)|)^\ell \leq (1 + |x|^\ell)$$

for some K . From (2.10), (3.1) we get

$$J(x, t, u) \leq M_1(1 + |x|^\rho), \quad \rho = \max(m+1, \ell)$$

for some M_1 . By part (a) of the Verification Theorem, $S(x, t) \leq J(x, t, u)$, which implies (i) when $m > 1$.

For $t > t_0 > 0$, $|\eta(t)|$ is bounded by a constant not depending on $x = \eta(0)$. Since $\xi(t) = \eta(t) + \zeta(t)$, and $E_x |\zeta(t)|^\ell$ is bounded, this bounds $E_x S^0[\xi(t)]$ by a constant not depending on x . The estimates above and part (a) of the Verification Theorem then give (ii).

It remains to prove (i) when $m = 1$. Consider the "trivial" control $u(\tau) \equiv 0$. When $m = 1$, g grows at most linearly and V at most quadratically as $|x| \rightarrow \infty$. Moreover, $E \|\xi\|_t^2 \leq K(1 + |x|^2)$ for some K . Using again (a) of the Verification Theorem, we get again (i) with $\rho = \max(2, \ell)$. [When $m = 1$, this is a known result, obtained without using stochastic control arguments.]

4. An existence theorem. In this section we give a stochastic control proof of a theorem asserting that the dynamic programming equation (2.2) with

the initial data S^0 has a solution S . The argument is essentially taken from [4, p. 222 and top p. 223.] Since (2.2) is equivalent to the linear equation (2.1), with positive initial data p^0 , one could get existence of S from other results which give existence of positive solutions to (2.1), see [10][11]. However, the stochastic control proof gives a polynomial growth condition on S used in the Verification Theorem (§2).

Let $0 < \alpha \leq 1$. We say that a function ϕ with domain Q is of class C_α if the following holds. For any compact $\Gamma \subset Q$, there exists M such that $(x,t), (x',t') \in \Gamma$ imply

$$(4.1) \quad |\phi(x',t') - \phi(x,t)| \leq M[|t'-t|^{\alpha/2} + |x'-x|^\alpha].$$

We say that ϕ is of class $C_\alpha^{2,1}$ if $\phi, \phi_{x_i}, \phi_{x_i x_j}, \phi_t$ are of class C_α , $i, j=1, \dots, n$.

In this section the following assumptions are made. The matrix $\sigma(x)$ is assumed constant. By a change of variables in R^n we may take

$$(4.2) \quad \sigma = \text{identity}$$

For fixed t , $g(\cdot, t), V(\cdot, t)$ are of class C^1 on R^n , and $g, g_{x_i}, V, V_{x_i}, i = 1, \dots, n$, are of class C_α for some $\alpha \in (0, 1]$.

Moreover,

$$(4.3) \quad |g(x,t)| \leq \gamma_1 + \gamma_2 |x|^m, \quad m \geq 1,$$

with γ_2 small enough that (4.8) below holds. (If $g \in \mathcal{P}_\mu$ with $\mu < m$, then we can take γ_2 arbitrarily small.) We assume that

$$(4.4) \quad a_1 |x|^{2m} - a_2 \leq -V(x,t) \leq A(1 + |x|^{2m})$$

for some positive a_1, a_2, A and that

$$(4.5) \quad g_x \in \mathcal{P}_m, \quad V_x \in \mathcal{P}_{2m}.$$

We assume that $S^0 \in C^3 \cap \mathcal{P}_\ell$ for some $\ell \geq 0$, and

$$(4.6) \quad \lim_{|x| \rightarrow \infty} S^0(x) = +\infty$$

$$(4.7) \quad |S_x^0| \leq C_1 S^0 + C_2$$

for some positive C_1, C_2 .

Example. Suppose that $V(x,t) = -kV_0(x) + V_1(x,t)$ with $V_0(x)$ a positive, homogeneous polynomial of degree $2m$, $k > 0$, and $V_1(x,t)$ a polynomial in x of degree $\leq 2m-1$ with coefficients Hölder continuous functions of t . Suppose that $g(x,t)$ is a polynomial of degree $\leq m-1$ in x , with coefficients Hölder continuous in t , and $S^0(x)$ is a polynomial of degree ℓ satisfying (4.6). Then all of the above assumptions hold.

From (2.9), (4.2), $L = \frac{1}{2} |u-g|^2 - V$. If γ_2 in (4.3) is small enough, then

$$(4.8) \quad \beta_1(|u|^2 + |x|^{2m}) - \beta_2 \leq L(x,t,u) \leq B(1 + |u|^2 + |x|^{2m})$$

for suitable positive β_1, β_2, B . Moreover,

$$L_x = -g_x(u-g) - V_x,$$

$$|L_x| \leq \frac{1}{2} |u|^2 + |g_x|^2 + \frac{1}{2} |g|^2 + |V_x|,$$

where $|g_x|$ denotes the operator norm of g_x regarded as a linear transformation on R^n . From (4.3), (4.5), (4.8)

$$(4.9) \quad |L_x| \leq C_1 L + C_2$$

for some positive C_1, C_2 (which we may take the same as in (4.7).)

Theorem 4.1 Let $r = \max(2m, \ell)$. Then equation (2.2) with initial data $S(x, 0) = S^0(x)$ has a unique solution $S(x, t)$ of class $C_\alpha^{2,1} \cap \mathcal{P}_r$, such that $S(x, t) \rightarrow \infty$ as $|x| \rightarrow \infty$ uniformly for $0 \leq t \leq t_1$.

Proof. We follow [4, §5]. For $k = 1, 2, \dots$, let us impose the constraint $|u| \leq k$ on the feedback controls admitted as drifts in (2.7).

Let

$$(4.10) \quad S_k(x, t) = \min_{|u| \leq k} J(x, t; u).$$

Then S_k is a $C_\alpha^{2,1}$ solution to the corresponding dynamic programming equation

$$(4.11) \quad (S_k)_t = \frac{1}{2} \Delta S_k + H_k(x, t, (S_k)_x),$$

$$H_k(x, t, (S_k)_x) = \min_{|u| \leq k} [L(x, t, u) + (S_k)_x u].$$

The initial data are again $S_k(x, 0) = S^0(x)$. The minimum in (4.10) is attained by an admissible u_k^{op} . See [7, p. 172].

Now $S_1 \geq S_2 \geq \dots$; and S_k is bounded below since L and S^0 are bounded below by (4.6), (4.9). Let $S = \lim_{k \rightarrow \infty} S_k$. Let us show that $(S_k)_x$ is bounded uniformly for (x, t) in any compact set. Once this is established standard arguments in the theory of parabolic partial differential equations imply that $S \in C_\alpha^{2,1}$ and S satisfies (2.2). For $(S_k)_x$ there is the probabilistic representation

$$(4.12) \quad (S_k)_x(x, t) = E_x \left\{ \int_0^t L_x[\xi_k(\tau), t-\tau, u_k(\tau)] d\tau + S_x^0[\xi_k(t)] \right\},$$

where ξ_k is the solution to (2.7) with $u = u_k^{op}$, $\xi_k(0) = x$, and

$$u_k(\tau) = u_k^{op}(\xi_k(\tau), \tau).$$

This can be proved exactly as in [4, Lemma 3]. Another proof, based on differentiating (4.10) with respect to x_i , $i=1, \dots, n$, is given in [5, Lemma 5.3]. From (4.7), (4.9), (4.12)

$$|(S_k)_x(x, t)| \leq C_1 E_x \int_0^t L[\xi_k(\tau), t-\tau, u_k(\tau)] d\tau + S^0[\xi_k(t)] + C_2(t+1),$$

or since u_k^{op} is optimal

$$(4.13) \quad |(S_k)_x(x, t)| \leq C_1 S_k(x, t) + C_2(t+1).$$

Since $S_k(x, t)$ is bounded uniformly on compact sets, (4.12) gives the required bound for $|(S_k)_x|$ uniformly on compact sets.

For the "trivial" control 0, we have by (4.8) and $S^0 \in \mathcal{P}_\ell$

$$J(x, t, 0) \leq B_1(1 + E_x \|\tilde{\xi}\|_t^r), \quad r = \max(2m, \ell)$$

for suitable B_1 . When $u(\tau) \equiv 0$, $\sigma = I$, we have $\tilde{\xi}(\tau) = x + w(\tau)$.

For suitable M we have

$$S_k(x, t) \leq J(x, t, 0) \leq M(1 + |x|^r), \quad k = 1, 2, \dots.$$

Hence $S(x, t)$ satisfies the same inequality. Since S is bounded below, this implies $S \in \mathcal{P}_r$.

Let us show that $S(x, t) \rightarrow \infty$ as $|x| \rightarrow \infty$, uniformly for $0 \leq t \leq t_1$.

Since $S_k(x, t) = J(x, t; u_k^{op})$, (4.8) implies

$$S_k(x, t) \geq \beta_1 E_x \int_0^t [|u_k(\tau)|^2 + |\xi_k(\tau)|^{2m}] d\tau - \beta_2 t + ES^0[\xi_k(t)] .$$

Given $\lambda > 0$ there exists R_1 such that $|x| \geq R_1$ implies $S^0(x) \geq \lambda$, by (4.6). Let $R_2 > R_1$ and consider the events

$$A_1 = \{ ||\xi_k - x||_t \leq R_2 - R_1 \}$$

$$A_2 = \{ ||v_k||_t \geq \frac{1}{2} (R_2 - R_1) \} , \quad v_k(\tau) = \int_0^\tau u_k(\theta) d\theta$$

$$A_3 = \{ ||w||_t \geq \frac{1}{2} (R_2 - R_1) \}$$

with $|| \cdot ||_t$ the sup norm on $[0, t]$. Since

$$\xi_k(\tau) - x = v_k(\tau) + w(\tau), \quad 0 \leq \tau \leq t,$$

$A_1^c \subset A_2 \cup A_3$. For $R_2 - R_1$ large enough, $P(A_3) < \frac{1}{4}$ and hence

$P(A_1) + P(A_2) \geq \frac{3}{4}$. From Cauchy-Schwarz

$$\frac{1}{4} (R_2 - R_1)^2 P(A_2) \leq t E_x \int_0^t |u_k(\theta)|^2 d\theta .$$

Let $|x| \geq R_2$. On A_1 , $|\xi_k(t)| \geq R_1$ and hence $S^0[\xi_k(t)] \geq \lambda$. For $|x| \geq R_2$

$$S_k(x, t) \geq \frac{\beta_1}{4t} (R_2 - R_1)^2 P(A_2) + \lambda P(A_1) - (\beta_2 t + \beta_3)$$

with β_3 a lower bound for $S^0(x)$ on R^n . Since the right side does not

depend on k , S satisfies the same inequality. This implies that

$S(x, t) \rightarrow \infty$ as $|x| \rightarrow \infty$, uniformly for $0 \leq t \leq t_1$.

To obtain uniqueness, $p = \exp(-S)$ is a $C_{\alpha}^{2,1}$ solution of (2.1), with $p(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly for $0 \leq t \leq t_1$. Since $V(x, t)$ is bounded

above, the maximum principle for linear parabolic equations implies that $p(x,t)$ is unique among solutions to (2.1) with these properties, and with initial data $p(x,0) = p^0(x) = \exp [-S^0(x)]$. Hence, S is also unique, proving theorem 4.1.

It would be interesting to remove the restriction that $\sigma = \text{constant}$ made in this section.

5. A lower estimate for $S(x,t)$. To complement the upper estimates in Theorem 3.1, let us give conditions under which $S(x,t) \rightarrow \infty$ as $|x| \rightarrow \infty$ at least as fast as $|x|^{m+1}$, $m \geq 1$. This is done by establishing a corresponding exponential rate of decay to 0 for $p(x,t)$. In this section we make the following assumptions. We take $\sigma \in C^2$ with

$$(5.1) \quad \sigma, \sigma^{-1}, \sigma_{x_i} \text{ bounded, } \sigma_{x_i x_j} \in \mathcal{P}_r, \quad i, j=1, \dots, n,$$

for some $r > 0$. For each t , $g(\cdot, t) \in C^2$. Moreover,

$$(5.2) \quad g \in \mathcal{P}_\mu, \mu < m, \quad g_{x_i} \in \mathcal{P}_r, \quad g_{x_i x_j} \in \mathcal{P}_r,$$

and $g, g_{x_i}, g_{x_i x_j}$ are continuous on Q . For each t , $V(\cdot, t) \in C^2$.

Moreover, V satisfies (4.4),

$$(5.3) \quad V_{x_i} \in \mathcal{P}_r, \quad V_{x_i x_j} \in \mathcal{P}_r,$$

and $V, V_{x_i}, V_{x_i x_j}$ are continuous on Q . We assume that $p^0 \in C^2$

and that there exist positive β, M such that

$$(5.4) \quad \exp [\beta|x|^{m+1}] [p^0(x) + |p_{x_i}^0(x)| + |p_{x_i x_j}^0(x)|] \leq M$$

Theorem 5.1. Let $p(x,t)$ be a $C^{2,1}$ solution to (2.1) such that $p(x,t) \rightarrow 0$ as $|x| \rightarrow \infty$, uniformly for $0 \leq t \leq t_1$. Then there exists $\delta > 0$ such that $\exp[\delta|x|^{m+1}]p(x,t)$ is bounded on Q .

Proof. Let

$$\psi(x) = (1+|x|^2)^{\frac{m+1}{2}}, \quad \pi(x,t) = \exp [\delta\psi(x)]p(x,t).$$

Then π is a solution to

$$(5.5) \quad \pi_t = \frac{1}{2} \text{tr } a\pi_{xx} + \bar{g} \cdot \pi_x + \bar{V}\pi,$$

$$\bar{g} = g - \delta a\psi_x$$

$$\bar{V} = V - \delta g \cdot \psi_x + \frac{1}{2} (\delta^2 a\psi_x \cdot \psi_x - \delta \text{tr } a\psi_{xx})$$

Following an argument in [10], equation (5.5) with initial data

$\pi^0 = \exp(\delta\psi)p^0$ has for small enough $\delta > 0$ the probabilistic solution

$$(5.6) \quad \tilde{\pi}(x,t) = E_x \{ \pi^0[X(t)] \exp \int_0^t [\sigma^{-1} \bar{g} dw - \frac{1}{2} |\sigma^{-1} \bar{g}|^2 d\tau + \bar{V} d\tau] \},$$

where $X(t)$ satisfies

$$(5.7) \quad dX = \sigma[X(\tau)]dw, \quad \tau \geq 0,$$

with $X(0) = x$. In the integrands $\sigma^{-1} \bar{g}$ and \bar{V} are evaluated at $(X(\tau), \tau)$.

This solution $\tilde{\pi}$ is bounded and $C^{2,1}$. We sketch the proof of these facts

below. Then $\tilde{p} = \exp(-\delta\psi)\tilde{\pi}$ is a $C^{2,1}$ solution to (2.1), with initial

data p^0 , and with $\tilde{p}(x,t)$ tending to 0 as $|x| \rightarrow \infty$ uniformly for

$0 \leq t \leq t_1$. By the maximum principle, $\tilde{p} = p$ which implies that $\exp[\delta|x|^{m+1}]p$ is bounded on Q .

It remains to indicate why $\tilde{\pi}$ is a solution to (5.5) with the required properties. We have $\psi_{x_i} \in \mathcal{P}_m$, $\psi_{x_i x_j} \in \mathcal{P}_{m-1}$. By assumption V satisfies (4.4), a is bounded, and $g \in \mathcal{P}_\mu$, $\mu < m$. Hence, for δ small enough there exist positive \bar{a}_1, \bar{a}_2 such that

$$\bar{V}(x, t) \leq \bar{a}_2 - \bar{a}_1 |x|^{2m}.$$

Moreover, for some K

$$|\sigma^{-1}(x)g(x, t)|^2 \leq K(1 + |x|^{2\mu}), \quad \mu < m.$$

From these inequalities one can get a bound

$$E \left(\exp \int_0^t [\sigma^{-1} \bar{g} dw - \frac{1}{2} |\sigma^{-1} \bar{g}|^2 d\tau + \bar{V} d\tau] \right)^j \leq M_j$$

for any $j > 0$. This gives a uniform integrability condition from which one gets that $\tilde{\pi}$ is a bounded $C^{2,1}$ solution of (5.5) by the usual technique of differentiating (5.6) twice with respect to the components x_1, \dots, x_n of the initial state $x = X(0)$. This proves Theorem 5.1.

Since $S = -\log p$, we get by taking logarithms:

Corollary. For some positive δ, δ_1

$$(5.8) \quad S(x, t) \geq \delta |x|^{m+1} - \delta_1.$$

6. Connection with the pathwise filter equation. The generator A of the signal process in (1.1) satisfies for $\phi \in C^2$

$$A\phi = \frac{1}{2} \text{tr } a(x) \phi_{xx} + b(x) \cdot \phi_x.$$

The pathwise filter equation (1.5) for $p = p^y$ is

$$(6.1) \quad p_t = (A^y)^* p + \tilde{V}^y p, \text{ where}$$

$$A^y \phi = A\phi - y(t) a(x) h_x(x) \cdot \phi_x$$

$$\tilde{V}^y(x, t) = -\frac{1}{2} h(x)^2 - y(t) A h(x) + \frac{1}{2} y(t)^2 h_x(x) \cdot a(x) h_x(x).$$

Hence, in (1.5) we should take

$$(6.2) \quad g^y = -b + y(t) a h_x + \gamma, \quad \gamma_j = \sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_j}, \quad j = 1, \dots, n,$$

$$(6.3) \quad v^y = \tilde{V}^y - \text{div}(b - y(t) a h_x) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j},$$

To satisfy the various assumptions about $g = g^y$, $V = V^y$ made above, suitable conditions on σ , b , and h must be imposed. To obtain the local Hölder conditions needed in §4 we assume that $y(\cdot)$ is Hölder continuous on $[0, t]$. This is no real restriction, since almost all observation trajectories $y(\cdot)$ are Hölder continuous.

To avoid unduly complicating the exposition let us consider only the following special case. We take $\sigma = \text{identity}$, an assumption already made for the existence theorem in §4. We assume that $b \in C^3$ with b , b_x bounded, and all second, third order partial derivatives of b of class \mathcal{P}_r for some r . Let h be a polynomial of degree m and S^0 a polynomial of degree ℓ , with

$$(6.4) \quad \lim_{|x| \rightarrow \infty} |h(x)| = \infty, \quad \lim_{|x| \rightarrow \infty} S^0(x) = +\infty.$$

Then all of the hypotheses in §'s 2-4 hold. In (6.2), g^y has polynomial growth of degree $m-1$ as $|x| \rightarrow \infty$, while in (6.3) v^y is the sum of the degree $2m$ polynomial $-\frac{1}{2}h^2(x)$ and terms with polynomial growth of degree $< 2m$.

Let $S^y = -\log p^y$. From Theorem 3.1 we get the upper bounds

$$(i) \quad S^y(x,t) \leq M_1(1+|x|^\rho), \quad 0 \leq t \leq t_1, \quad \rho = \max(m+1, \ell)$$

$$(ii) \quad S^y(x,t) \leq M_2(1+|x|^{m+1}), \quad 0 < t_0 \leq t \leq t_1, \quad m > 1$$

where M_1, M_2 depend on y . For $p^0 = \exp(-S^0)$ to satisfy (5.4) we need $\ell \geq m+1$. The Corollary to Theorem 5.1 then gives the lower bound

$$(6.6) \quad S^y(x,t) \geq \delta|x|^{m+1} - \delta_1, \quad 0 \leq t \leq t_1.$$

From (6.5)(ii) and (6.6) we see that $S^y(x,t)$ increases to $+\infty$ like $|x|^{m+1}$, at least for $m > 1$ and t bounded away from 0, and for $0 \leq t \leq t_1$, in case $\ell = m+1$.

Finally, $q = \exp(y(t)h)p$ is a solution to the Zakai equation. For any $\phi \in C_b$ (i.e., ϕ continuous and bounded on R^n) let

$$\begin{aligned} \tilde{\Lambda}_t(\phi) &= \int_{R^n} \phi(x) q(x,t) dx, \\ \Lambda_t(\phi) &= E^0 \left\{ \phi[x(t)] \exp \int_0^t (h[x(\tau)] dy - \frac{1}{2} h[x(\tau)]^2 d\tau) \middle| \mathcal{F}_t(y) \right\}, \end{aligned}$$

where E^0 denotes expectation with respect to the probability measure P^0 obtained by eliminating the drift term in (1.2) by a Girsanov transformation. The measure Λ_t is the unnormalized conditional distribution of $x(t)$.

By a result of Sheu [10] $\Lambda_t = \tilde{\Lambda}_t$ and hence $q(\cdot, t)$ is the density of Λ_t .

In fact, both Λ_t , $\tilde{\Lambda}_t$ are weak solutions to the Zakai equation. Moreover,

$$E\Lambda_t(1) = 1, E\tilde{\Lambda}_t(1) \leq 1.$$

The inequality is seen by approximating h by bounded h_k , with corresponding density $q_k(x, t)$ of the unnormalized conditional distribution

Λ_{kt} . Then (see [10])

$$E\Lambda_{kt}(1) = 1, \Lambda_{kt}(\phi) \rightarrow \tilde{\Lambda}_t(\phi) \text{ as } k \rightarrow \infty,$$

for any continuous ϕ with compact support. Hence, $E\tilde{\Lambda}_t(1) \leq 1$. The uniqueness theorem in [10] for weak solutions to the Zakai equation implies $\Lambda_t = \tilde{\Lambda}_t$.

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